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A projection based approach to the Clebsch–Gordan multiplicity problem for compact semisimple Lie groups: III. The classical limit

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Abstract. This paper is the third in a series directed towards a projection-based solution to the Clebsch-Gordan multiplicity problem for semisimple (compact) Lie groups. In this paper it is shown that the projected states for the Clebsch-Gordan problem approach orthogonality in the classical limit of large quantum numbers in a manner analogous to that of Elliott's well known solution to the U(3) ⊃ O(3) state labelling problem.

1. Introduction

In two previous papers (Edwards and Gould 1986a, b, hereafter referred to as I and II respectively) a projection-based solution to the Clebsch-Gordan problem for a general semisimple (compact) Lie group was proposed. The principal motivation for this work was provided by Elliott's approach (Elliott 1958, Elliott and Harvey 1962) to the O(3) ⊂ U(3) state labelling problem. In this latter model the choice of intrinsic states, from which projection is done, is physically and mathematically natural, directly related to the fact that, in the appropriate realisation of the embedding of O(3) in U(3) and choice of Cartan subalgebra of U(3), a highest weight vector of SU(3) is cyclic under the action of SO(3) for the irreducible representation of SU(3) in which it lies. These desirable features of the O(3) ⊂ U(3) multiplicity problem—the existence, for a given choice of Cartan subalgebra, of mathematically distinguished cyclic vectors for the representations to be reduced, and the availability of a natural choice of intrinsic states from which to project—are shared by the Clebsch-Gordan problem for a semisimple Lie group G as demonstrated in I. The details of these results (and additional properties) are discussed in detail in I and II.

The approach of I and II to the Clebsch-Gordan problem suffers from the drawback, which is typical of projection methods, that the projected states are non-orthogonal. It is our aim in this paper to demonstrate that the semisimple (compact) Lie group Clebsch-Gordan problem possesses the desirable feature that the projected states approach orthogonality in a suitable limit of large quantum numbers. Our motivation for this work is again provided by Elliott's (1958) solution to the O(3) ⊂ U(3) state labelling problem in which the projected O(3) states rapidly approach orthogonality as the SU(3) highest weight labels are increased (Draayer *et al* 1968, Elliott 1958,

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Elliott and Harvey 1962). A related approach has been considered by Biedenharn and co-authors (Louck and Biedenharn 1972, Lohe *et al* 1977, 1983) for the U(n) tensor operator problem, who consider the behaviour of certain coupling coefficients in various limits.

It was shown in I that the overlap coefficients between the projected states are reasonably well behaved since they determine rational polynomial functions on the dual of the Cartan subalgebra. This property is particularly convenient for discussing the asymptotic behaviour of the projected states in a suitable limit of large quantum numbers. It shall be demonstrated that asymptotically the projected states for the Clebsch-Gordan problem approach the intrinsic states from which projection is done. In particular the solution to the Clebsch-Gordan problem, as proposed in I and II, is asymptotically orthogonal.

2. Preliminaries

Throughout we adopt the notation and conventions of I and II. We assume that L is a semisimple Lie algebra of rank l with universal enveloping algebra U and H is a fixed Cartan subalgebra (CSA) of L. We let $V(\lambda)$ be a fixed (but arbitrary) finite-dimensional irreducible U module with highest weight $\lambda \in \Lambda^+$. We let $\lambda_1, \ldots, \lambda_m$ denote the *distinct* weights occurring in $V(\lambda)$ with multiplicities $n_1, \ldots, n_m$ resp. We assume that the weights are in non-decreasing order with respect to the usual partial ordering $>$ induced on the weights by the positive roots (Humphreys 1972): i.e. if $\lambda_i > \lambda_j$ then $i > j$. With this convention we have $\lambda_m = \lambda$, the highest weight of $V(\lambda)$, and $\lambda_1 = -\lambda^*$ (λ^* the highest weight of the dual module $V(\lambda)^*$) which is the lowest weight in $V(\lambda)$. For each $i(=1, \ldots, m)$ we let $V_i(\lambda)$ denote the space of weight vectors of weight λ_i. We clearly have $n_i = \dim V_i(\lambda)$.

Now let $V(\mu)$ be an arbitrary finite-dimensional irreducible U module with highest weight $\mu \in \Lambda^+$. Then the Clebsch-Gordan (CG) reduction of the tensor product module $V(\lambda) \otimes V(\mu)$ may be formally written (I, II)

$$V(\lambda) \otimes V(\mu) = \bigoplus_{i=1}^{m} m(\mu + \lambda_i \colon \lambda \otimes \mu) V(\mu + \lambda_i) \qquad \mu + \lambda_i \in \Lambda^+$$

where the multiplicities are given by

$$m(\mu + \lambda_i \colon \lambda \otimes \mu) = \dim V_{i,\mu}(\lambda)$$

where

$$V_{i,\mu}(\lambda) = \{v \in V_i(\lambda) | x_j^{\langle \mu + \delta, \alpha_j \rangle} v = 0; j = 1, \ldots, l\}. \tag{1}$$

In particular the multiplicities (possibly zero) must satisfy

$$0 \leq m(\mu + \lambda_j \colon \lambda \otimes \mu) \leq n_i.$$

Now let $\{e_{i,j} | j = 1, \ldots, m(\mu + \lambda_i \colon \lambda \oplus \mu)\}$ be a basis for the space $V_{i,\mu}(\lambda)$ of (1) and let e_+^μ denote the maximal weight vector of $V(\mu)$. From I and II we know that a full set of independent maximal weight states of weight $\mu + \lambda_i$ is given by the vectors

$$P_i | e_{i,j} \otimes e_+^\mu \rangle \qquad j = 1, \ldots, m(\mu + \lambda_i \colon \lambda \otimes \mu) \tag{2}$$

where P_i is the (central) projection onto submodules $V(\mu+\lambda_i)$. This method of solution to the CG state labelling problem, which has been discussed in detail in I and II, suffers from the drawback that the states (2) are not generally orthogonal, i.e. the overlap coefficients

$$\langle e_{i,j}\otimes e_+^\mu|P_i|e_{i,k}\otimes e_+^\mu\rangle \qquad k\neq j$$

are not generally zero. Nevertheless the overlap coefficients are relatively well behaved as can be seen from the following result derived in I.

Theorem 1. Let $\{e_j^i\}_{j=1}^{n_i}$ be a basis for the weight space $V_i(\lambda)$. Then the overlap coefficients

$$\langle e_k^i\otimes e_+^\mu|P_i|e_j^i\otimes e_+^\mu\rangle \qquad k,j=1,\ldots,n_i \tag{3}$$

determine functions of $\mu\in\Lambda^+$ which may be extended uniquely to rational polynomial functions on all of H^*.

This rational polynomial function property of the overlap coefficients is useful and in particular indicates that it may be possible to evaluate them analytically. Moreover it was noted in paper I that the rational polynomial functions (3) are uniquely determined by their values in the full multiplicity region which considerably simplifies the task of evaluation. It is our aim in this paper to investigate the asymptotic behaviour of these rational polynomial functions in the large quantum number limit $\langle\mu,\alpha_i\rangle\to\infty$ $(i=1,\ldots,l)$.

Following Kostant (1975) we say that λ is *subordinate* to μ if $\mu+\lambda_i\in\Lambda^+(i=1,\ldots,m)$. In such a case all irreducible modules $V(\mu+\lambda_i)$ occur in $V(\lambda)\otimes V(\mu)$ with full multiplicity $m(\mu+\lambda_i\colon\lambda\otimes\mu)=n_i$. It is convenient to introduce positive integers $d_i(i=1,\ldots,l)$ defined to be smallest with respect to the property

$$x_i^{d_i+1}v=0 \qquad \forall v\in V(\lambda).$$

The integers d_i have been determined by Kostant (1959) and Fiengold (1978). It follows, in view of (1), that λ is subordinate to μ if and only if

$$\langle\mu,\alpha_i\rangle\geqslant d_i \qquad i=1,\ldots,l. \tag{4}$$

The set of weights $\mu\in\Lambda^+$ satisfying the inequalities (4) constitute the full multiplicity region in Λ^+.

The tensor product module $V(\lambda)\otimes V(\mu)$ is known (Gould and Edwards 1984) to be cyclically generated by the vector $e_-^\lambda\otimes e_+^\mu$ where e_-^λ (resp e_+^μ) is the minimal (resp maximal) weight vector of $V(\lambda)$ (resp $V(\mu)$). We conclude this section with the following result (Gould and Edwards 1984) on finite-dimensional cyclic modules.

Theorem 2. Let $V=Uv^\mu$ be a finite-dimensional cyclic module generated by a vector of weight $\mu\in\Lambda$. Set $V^{(0)}=U(B)v^\mu$ and let $\pi^{(0)}$ denote the set of *distinct* weights in $V^{(0)}$. For $\nu\in\pi^{(0)}$ let $m_0(\nu)$ denote the multiplicity of the weight ν in $V^{(0)}$. Suppose

$$V=\bigoplus_\lambda V(\lambda) \tag{5}$$

is the decomposition of V into irreducible submodules. Then the following hold.

(a) The highest weights occurring in the decomposition (5) are of the form $\lambda\in\pi^{(0)}\cap\Lambda^+$. In particular $\lambda=\mu$ or $\lambda>\mu$.

(b) The irreducible module $V(\lambda)$ occurs in V with multiplicity $m_v(\lambda) \leq m_0(\lambda)$. In particular $m_v(\mu) \leq 1$.

(c) $m_v(\mu) = 1$ if and only if $v^\mu \notin \Sigma_{i=1}^{l} Ux_i v^\mu$.

Remarks. We note that in the above theorem we have adopted the notation of Gould and Edwards (1984) where B denotes the nilpotent subalgebra of L generated by the raising generators $x_\alpha \in L_\alpha$ corresponding to positive roots $\alpha \in \Phi^+$, and U(B) is the universal enveloping algebra of B.

3. The large quantum number limit

In this section we investigate the CG problem for the tensor product module $V(\lambda) \otimes V(\mu)$ in the asymptotic limit, i.e. we assume $\langle \mu, \alpha_i \rangle \gg 1 (i = 1, \ldots, l)$. We find it convenient to introduce a small parameter $\varepsilon \geq 0$ of the same order as the numbers $1/\langle \mu, \alpha_i \rangle (i = 1, \ldots, l)$. Explicitly we may define ε according to

$$\varepsilon = \max_i \varepsilon_i \qquad \varepsilon_i = 1/\langle \mu, \alpha_i \rangle \qquad i = 1, \ldots, l.$$

Then for $\alpha \in \Phi^+$ arbitrary, we have that $1/\langle \mu, \alpha \rangle$ is of *order* $\varepsilon(\mathrm{O}(\varepsilon))$.

For such *large* $\mu \in \Lambda^+$ we are considering we may assume that λ is subordinate to μ (i.e. μ satisfies the inequalities of (4)) and hence all irreducible modules $V(\mu + \lambda_i)$ $(i = 1, \ldots, m)$ occur in $V(\lambda) \otimes V(\mu)$ with full multiplicity n_i. We may thus write

$$V(\lambda) \otimes V(\mu) = \bigoplus_{i=1}^{m} n_i V(\mu + \lambda_i). \tag{6}$$

We denote the (central) projection onto the primary submodule $n_i V(\mu + \lambda_i)$ by P_i. Now let $\{e_j^i\}_{j=1}^{n_i}$ constitute an orthonormal basis for the weight space $V_i(\lambda)$. Then it follows from § 2 (see I and II) that the vectors

$$P_i | e_j^i \otimes e_+^\mu \rangle \qquad j = 1, \ldots, n_i \tag{7}$$

constitute a full set of (non-orthogonal) highest weight states of weight $\mu + \lambda_i$. It is our aim here to demonstrate that the states (7) are orthogonal to first order in ε.

We note that the projection operator P_i may be regarded as a $D[\lambda] \times D[\lambda]$ $(D[\lambda] = \dim V(\lambda))$ matrix of operators with entries

$$\langle e_{j'}^{k'} | P_i | e_j^k \rangle \in \mathrm{End}\ V(\mu)$$

defined by

$$\langle e_\alpha^\mu | \langle e_{j'}^{k'} | P_i | e_j^k \rangle | e_\beta^\mu \rangle = \langle e_{j'}^{k'} \otimes e_\alpha^\mu | P_i | e_j^k \otimes e_\beta^\mu \rangle$$

where $e_\alpha^\mu, e_\beta^\mu$ are arbitrary vectors in $V(\mu)$. We define the λ trace of P_i according to

$$\tau_\lambda(P_i) = \sum_{k=1}^{m} \sum_{j=1}^{n_k} \langle e_j^k | P_i | e_j^k \rangle \in \mathrm{End}\ V(\mu).$$

We have the following result.

Lemma 1.

(a) $P_i | e_j^k \otimes e_+^\mu \rangle = 0$ for $k > i, j = 1, \ldots, n_k$.

(b) $\tau_\lambda(P_i)$ reduces to a scalar multiple ξ_i of the identity on $V(\mu)$. We have

$$\xi_i = n_i D[\mu + \lambda_i] / D[\mu] = n_i + \mathrm{O}(\varepsilon).$$

Proof.

(a) For $k=1,\ldots,m$ we have

$$\mathrm{U(B)}|e_j^k\otimes e_+^\mu\rangle=[\mathrm{U(B)}e_j^k]\otimes e_+^\mu\subseteq\bigoplus_{l=k}^m V_l(\lambda)\otimes e_+^\mu$$

where the last inclusion follows from the assumption that the weights $\lambda_1,\ldots,\lambda_m$ are in non-decreasing order. Hence it follows from theorem 2(a) that the highest weights occurring in the cyclic module generated by $e_j^k\otimes e_+^\mu$ are of the form $\mu+\lambda_l(l\geqslant k)$. In particular if $i<k$ it follows that the irreducible module $V(\mu+\lambda_i)$ cannot occur in $\mathrm{U}|e_j^k\otimes e_+^\mu\rangle$. This is enough to prove

$$P_i|e_j^k\otimes e_+^\mu\rangle=0 \qquad \text{for } i<k$$

as required.

As to (b) note that P_i intertwines the action of L, namely

$$[\pi_\lambda(x)\otimes 1+1\otimes\pi_\mu(x)]P_i=P_i[\pi_\lambda(x)\otimes 1+1\otimes\pi_\mu(x)] \qquad x\in \mathrm{L}.$$

Rearranging we may thus write

$$[\pi_\lambda(x)\otimes 1, P_i]=[P_i, 1\otimes\pi_\mu(x)].$$

Taking the λ trace of both sides of this equation we obtain, from the properties of trace,

$$[\tau_\lambda(P_i), \pi_\mu(x)]=\tau_\lambda[\pi_\lambda(x)\otimes 1, P_i]=0 \qquad x\in\mathrm{L}.$$

Thus, from Schur's lemma, $\tau_\lambda(P_i)$ reduces to a scalar multiple ξ_i of the identity on $V(\mu)$.

Now let $\tau_{\lambda\otimes\mu}$ denote the total trace of P_i on the space $V(\lambda)\otimes V(\mu)$. We have immediately

$$\tau_{\lambda\otimes\mu}(P_i)=n_iD[\mu+\lambda_i]. \tag{8}$$

On the other hand the total trace is related to the partial traces τ_λ, τ_μ with respect to the spaces $V(\lambda)$ and $V(\mu)$ respectively according to

$$\tau_{\lambda\otimes\mu}(P_i)=\tau_\mu[\tau_\lambda(P_i)]=\xi_iD[\mu].$$

Comparing with (8) we thus obtain

$$\xi_i=n_i(D[\mu+\lambda_i]/D[\mu])$$

as required. Finally using Weyl's dimension formula (Humphreys 1972) we may write

$$\begin{aligned}\xi_i&=n_i\prod_{\alpha\in\Phi^+}\frac{\langle\mu+\lambda_i+\delta,\alpha\rangle}{\langle\mu+\delta,\alpha\rangle}\\&=n_i\prod_{\alpha\in\Phi^+}\left(1+\frac{\langle\lambda_i,\alpha\rangle}{\langle\mu+\delta,\alpha\rangle}\right)=n_i+\mathrm{O}(\varepsilon).\end{aligned}$$

We are now in a position to prove our main result.

Theorem 3.

$$\langle e_{j'}^{k'}\otimes e_+^\mu|P_i|e_j^k\otimes e_+^\mu\rangle=\delta_{k'i}\delta_{ik}\delta_{j'j}+\mathrm{O}(\varepsilon). \tag{9}$$

Proof. We proceed by recursion on $i(=1,\ldots,m)$ starting with $i=1$. For this case we have $n_i=1$ and e_1^1 is the (unique) minimal weight vector of $V(\lambda)$. Then we have from lemma 1(b)

$$\langle e_+^\mu|\tau_\lambda(P_1)|e_+^\mu\rangle=\sum_{k,j}\langle e_j^k\otimes e_+^\mu|P_1|e_j^k\otimes e_+^\mu\rangle$$

$$=\frac{D[\mu+\lambda_1]}{D[\mu]}=1+\mathrm{O}(\varepsilon).$$

However, from lemma 1(a) we have $P_1|e_j^k\otimes e_+^\mu\rangle=0$ for $k>1$ whence

$$\langle e_1^1\otimes e_+^\mu|P_1|e_1^1\otimes e_+^\mu\rangle=\langle e_+^\mu|\tau_\lambda(P_1)|e_+^\mu\rangle=1+\mathrm{O}(\varepsilon).$$

Thus we must have

$$\langle e_{j'}^{k'}\otimes e_+^\mu|P_1|e_j^k\otimes e_+^\mu\rangle=\delta_{k'1}\delta_{1k}\delta_{j'j}+\mathrm{O}(\varepsilon)$$

which proves the result for $i=1$.

Proceeding recursively assume the result holds for i. We prove the result for P_{i+1}. By the recursion hypothesis we have

$$\langle e_j^k\otimes e_+^\mu|P_k|e_j^k\otimes e_+^\mu\rangle=1+\mathrm{O}(\varepsilon)\qquad \text{for } k\leqslant i$$

which is equivalent to writing

$$|e_j^k\otimes e_+^\mu\rangle=P_k|e_j^k\otimes e_+^\mu\rangle+\alpha\omega$$

where ω is a normalised vector in $V(\lambda)\otimes V(\mu)$ and α is of order ε. Thus we have, for $k\leqslant i$,

$$P_{i+1}|e_j^k\otimes e_+^\mu\rangle=\alpha P_{i+1}\omega$$

which implies

$$\langle v|P_{i+1}|e_j^k\otimes e_+^\mu\rangle=\mathrm{O}(\varepsilon)\qquad \forall k\leqslant i \tag{10}$$

where v is an arbitrary normalised vector in $V(\lambda)\otimes V(\mu)$. We already know from lemma 1(a) that

$$P_{i+1}|e_j^k\otimes e_+^\mu\rangle=0\qquad \text{for } k<i+1.$$

Thus to prove our result it remains to demonstrate

$$\langle e_{j'}^{i+1}\otimes e_+^\mu|P_{i+1}|e_j^{i+1}\otimes e_+^\mu\rangle=\delta_{j',j}+\mathrm{O}(\varepsilon)\qquad (j,j'=1,\ldots,n_{i+1}). \tag{11}$$

We have, from lemma 1(b)

$$\langle e_+^\mu|\tau_\lambda(P_{i+1})|e_+^\mu\rangle=n_{i+1}+\mathrm{O}(\varepsilon).$$

Now the LHS of the above equation may be written

$$\langle e_+^\mu|\tau_\lambda(P_{i+1})|e_+^\mu\rangle$$

$$=\sum_{k,j}\langle e_j^k\otimes e_+^\mu|P_{i+1}|e_j^k\otimes e_+^\mu\rangle$$

$$=\sum_{j=1}^{n_{i+1}}\langle e_j^{i+1}\otimes e_+^\mu|P_{i+1}|e_j^{i+1}\otimes e_+^\mu\rangle+\sum_{k<i+1}\sum_{j=1}^{n_k}\langle e_j^k\otimes e_+^\mu|P_{i+1}|e_j^k\otimes e_+^\mu\rangle$$

$$+\sum_{k>i+1}^{m}\sum_{j=1}^{n_k}\langle e_j^k\otimes e_+^\mu|P_{i+1}|e_j^k\otimes e_+^\mu\rangle.$$

By lemma 1(a) the third term on the RHS above vanishes and the second term, by (10), is of order ε. Thus we must have

$$\sum_{j=1}^{n_{i+1}} \langle e_j^{i+1} \otimes e_+^\mu | P_{i+1} | e_j^{i+1} \otimes e_+^\mu \rangle = n_{i+1} + \mathrm{O}(\varepsilon).$$

However, since P_{i+1} is a projection, we must have

$$0 \leqslant \langle e_j^{i+1} \otimes e_+^\mu | P_{i+1} | e_j^{i+1} \otimes e_+^\mu \rangle \leqslant 1$$

which implies that

$$\langle e_j^{i+1} \otimes e_+^\mu | P_{i+1} | e_j^{i+1} \otimes e_+^\mu \rangle = 1 + \mathrm{O}(\varepsilon) \qquad j = 1, \ldots, n_{i+1}.$$

Thus we may write

$$P_{i+1} | e_j^{i+1} \otimes e_+^\mu \rangle = | e_j^{i+1} \otimes e_+^\mu \rangle + \alpha\omega$$

where $\omega \in V(\lambda) \otimes V(\mu)$ is normalised and α is of order ε. This is enough to establish (11) as required, which proves the final result by (finite) recursion.

The above result demonstrates that the projected states of (7) are approximated (in the sense of least squares) by the states $|e_j^i \otimes e_+^\mu\rangle$ to order ε. In particular the states (7) approach orthogonality in the limit of large quantum numbers. This asymptotic orthogonality is clearly analogous to the asymptotic orthogonality satisfied by Elliot's well known solution (Draayer *et al* 1968, Elliott 1958, Elliott and Harvey 1962) to the $\mathrm{U}(3) \supset \mathrm{O}(3)$ state labelling problem.

In terms of overlap coefficients, we let $f_{j,k}^i(\mu)(j, k = 1, \ldots, n_i)$ denote the rational polynomial functions on H* determined by the overlap coefficients

$$f_{j,k}^i(\mu) = \langle e_j^i \otimes e_+^\mu | P_i | e_k^i \otimes e_+^\mu \rangle \qquad \mu \in \Lambda^+.$$

From theorem (3) we have the limit property

$$\lim_{\langle \mu, \alpha_r \rangle \to \infty} f_{k,j}^i(\mu) = \delta_{kj} \qquad r = 1, \ldots, l \tag{12}$$

provided the limit is taken in Λ^+ which is equivalent to taking the limit in Λ (the set of integral linear functions on H*). However, Λ is Zariski dense in H* and polynomial functions are continuous in the Zariski topology on H* (Humphreys 1972). By continuity it follows therefore, since $f_{j,k}^i(\mu)$ determines a rational polynomial function, that the limit (12) holds in H* (and not just Λ^+). In conclusion we note that the limit (12) is well defined since it is independent of how the limit is reached.

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